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The 1/2 BPS 't Hooft loops in $\mathcal{N} = 4$ SYM as instantons in 2d Yang-Mills

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Abstract

We extend the recent conjecture on the relation between a certain 1/8 BPS subsector of 4d $\mathcal{N} = 4$ SYM on S^2 and 2d Yang-Mills theory by turning on circular 1/2 BPS 't Hooft operators linked with S^2. We show that localization predicts that these 't Hooft operators and their correlation functions with Wilson operators on S^2 are captured by instanton contributions to the partition function of the 2d Yang-Mills theory. Based on this prediction, we compute explicitly correlation functions involving the 't Hooft operator, and observe precise agreement with S-duality predictions.

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Contents

1 Introduction

The $\mathcal{N} = 4$ Super Yang-Mills theory is believed to enjoy an exact quantum symmetry, known as S-duality [1, 2, 3, 4], which relates weak coupling to strong coupling physics, and can be thought of as a non-abelian generalization of the familiar electric-magnetic duality of Maxwell's theory. More precisely, the S-duality symmetry of $\mathcal{N} = 4$ SYM with gauge group G acts on the complex coupling $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{4d}^2}$ as

$$\tau \to {}^L\tau = -\frac{1}{n_{\mathfrak{g}}\tau}\,, \tag{1.1}$$

and exchanges the gauge group G with its S-dual, or *Langlands dual*, group ${}^LG$ [2, 5, 6, 7]. Here $\mathfrak{g}$ denotes the Lie algebra of G, and $n_{\mathfrak{g}} = 1$ for the simply laced Lie algebras, $n_{\mathfrak{g}} = 2$ for $\mathfrak{so}(2N+1)$, $\mathfrak{sp}(N)$, $\mathfrak{f}_4$ and $n_{\mathfrak{g}} = 3$ for $\mathfrak{g}_2$. The transformation (1.1), together with the elementary symmetry $\tau \to \tau + 1$, generate an infinite group Γ which is a discrete subgroup of $SL(2, \mathbb{R})$. For the simply laced Lie algebras, this is just the familiar $SL(2, \mathbb{Z})$ modular group acting on τ.

Under S-duality, the electric and magnetic degrees of freedom are mapped into each other. In particular, the Wilson loop operator, which describes an electric charge running along a contour in space-time, should be mapped to its magnetic counterpart, the 't Hooft loop. In a gauge theory with a gauge group G, a Wilson loop operator is defined as the holonomy of the gauge field along a given contour (or, in supersymmetric

theories, as a suitable generalization involving scalar fields), and hence is labeled by a choice of representation R of the gauge group G. On the other hand, a 't Hooft operator cannot be described as a functional of the fields, but rather is defined by requiring that in the path integral we integrate over configurations such that the the gauge field (and scalars in the supersymmetric case) have a prescribed monopole-like singularity along the given contour. It can be seen that 't Hooft operators in a theory with gauge group G are labeled by representations ${}^L R$ of the *dual* group ${}^L G$ [2][8][5]. According to S-duality, the Wilson loop $W_R(\mathcal{C})$ in the theory with gauge group G is then mapped to the 't Hooft loop $T_R(\mathcal{C})$ in the theory with gauge group ${}^L G$, inserted along the same contour $\mathcal{C}$ and labeled by the representation R of G, and vice versa. In particular, their quantum expectation values are supposed to be equal upon the replacement (1.1). The following table summarizes the action of S-duality on Wilson and 't Hooft operators:

S-duality

Gauge group G	Gauge group ${}^L G$
τ	${}^L \tau$
't Hooft loop in rep ${}^L R$ of ${}^L G$	Wilson loop in rep ${}^L R$ of ${}^L G$
Wilson loop in rep R of G	't Hooft loop in rep R of G

Because S-duality relates weak and strong coupling dynamics, it is in general hard to perform explicit quantitative tests of the conjecture. However, non-trivial confirmation of the duality may be obtained by studying loop operators which preserve some fractions of the supersymmetries of the theory (in general, combinations of ordinary and superconformal supersymmetries). In this situation, one may in fact be able to obtain *exact* results for their quantum correlation functions, interpolating between weak and strong coupling. The best known example is the 1/2 BPS circular Wilson loop which couples to one of the six scalars field. The expectation value of this operator is exactly captured by a simple Gaussian matrix model, as first conjectured in [9][10] and proved in [11] using localization for the gauge theory on S^4.

Generalizing upon this example, a new large class of supersymmetric Wilson loops has been constructed in [12, 13]. These operators are defined for arbitrary contours on a round S^3 in space-time and couple to three of the six scalars. Generically, they are 1/16 BPS. A rather interesting sub-family can be defined by restricting the contours to lie on a great S^2 inside S^3. The corresponding loop operators are 1/8 BPS and they were conjectured to be exactly captured by the "zero-instanton sector" of 2d Yang-Mills theory on S^2 [13, 14]. This is in turn related to simple Gaussian matrix models with area dependent couplings [15, 16]. The 1/2 BPS circular loop is consistently recovered as a special case, and corresponds to an equator of the S^2. Several evidences in favor

of the conjecture, both from perturbation theory and from the dual string theory in $AdS_5 \times S^5$, have been presented in [13, 14, 17, 18, 19, 20].

In [21], extending the results of [11] to the case of the 1/8 BPS loops on S^2, the localization framework for the gauge theory on S^4 was used to argue that, for smooth field configurations, the 4d path integral localizes to a 2d field theory which turns out to be closely related to the Yang Mills Hitchin/Higgs theory (YMH) [22, 23, 24, 25]. For the purpose of computing correlation functions of the 1/8 BPS Wilson loops on S^2, this theory was argued in [21] to be perturbatively equivalent to pure 2d Yang-Mills theory, and also a natural explanation for the absence of non-trivial 2d instanton contributions (based on the appearance of extra fermion zero modes) was given. The explicit computation of the one-loop determinant for fluctuations normal to the localization locus was left open in [21], but there are reasons to believe that it could be trivial as in the 1/2 BPS case [11], hence the results of [21] would essentially support the conjecture of [13, 14].

In fact, the localization framework of [21] turns out to be rather rich and allows one to establish a more general dictionary between physical observables of the 4d theory which share some supersymmetry with the 1/8 BPS loops and observables of the 2d theory on S^2. An example is given by certain local chiral primary operators which can be inserted at arbitrary points on S^2: on the 2d theory side, they correspond to insertions of powers of the 2d YM field strength, and exact results for mixed correlation functions of local and Wilson loop operators can be obtained from 2d YM [26].

Another interesting example, which is the main subject of this paper, is the case of the 1/2 BPS circular 't Hooft loop operator. By examining the supersymmetry equations of [21] which dictate the localization, one can realize that a 1/2 BPS 't Hooft loop inserted along a great circle of S^4 linked to the S^2 on which the Wilson loops live is also Q-closed, where Q denotes the supercharge used in the localization (one of the four supercharges preserved by the Wilson loops). In other words, the 't Hooft loop is a particular solution of the supersymmetry equations with a monopole singularity at the center of a solid ball whose boundary is the interesting S^2. To rigorously understand how localization works in the presence of the magnetic loop, one should study the full moduli space of solutions of the supersymmetry equations in the presence of the singularity, generalizing the analysis of [21] where smooth field configurations were assumed. In this paper we do not perform this analysis, and instead propose a natural conjecture based on the following simple observation: the classical field configuration generated by the 't Hooft loop, when restricted to the S^2, is precisely equivalent to the (unstable) instanton solution of 2d YM labeled by the same quantum numbers of the 't Hooft loop, i.e. a representation ${}^LR$ of the dual group ${}^LG$. Hence we propose that, in the presence of the 1/2 BPS 't Hooft loop, the 4d path integral, with possible

insertions of Q-closed observables, localizes to the path integral of 2d YM around non-trivial unstable instantons. In the case of the minuscule representations[2] there are no complications related to subleading corrections, or so-called "monopole bubbling" [5, 27, 28]. In this case, we conjecture that the 't Hooft loop with highest weight ${}^L\lambda$ is captured by the contribution to the 2d Yang-Mills partition function of the unstable instanton labeled by ${}^L\lambda$. For general representations, we expect contributions of instantons associated with shorter weights appearing in that representation. In this paper, we mainly concentrate on the simplest case $G = U(N)$, for which ${}^LG = G$. Also, here we restrict to the case of imaginary τ (i.e. $\theta = 0$). We leave the study of more general gauge groups and representations, as well as non-zero θ, to future work.

According to S-duality, the expectation value of the 1/2 BPS 't Hooft loop should be given by the same Gaussian matrix model which captures the 1/2 BPS Wilson loop [9, 10, 11], with an inverted coupling constant as given by (1.1). Recently, this expectation was shown to be consistent with perturbation theory in [27], where a direct one-loop computation of the 't Hooft loop expectation value was carried out. In this paper, we apply our conjecture to obtain an exact prediction for the vev of the 1/2 BPS 't Hooft loop in $\mathcal{N} = 4$ SYM, and show that this is indeed precisely given by the Gaussian matrix model with dual coupling constant, as required by S-duality. Our conjecture also allows us to derive new exact predictions for correlation functions of the 't Hooft loop with any number of 1/8 BPS loops on S^2, by computing Wilson loop correlators in the 2d theory in the background of an instanton. As an example, we present the result for the correlator of the 't Hooft loop and the 1/2 BPS Wilson loop at the S^2 equator when both operators are labeled by the fundamental representation of $U(N)$, and show that the result is precisely consistent with S-duality.

There are several directions in which one may try to complete and extend the present work. In this paper we only consider the case when τ is purely imaginary, i.e. $\theta = 0$. So one immediate generalization is understand the case of general τ, as well as the dyonic Wilson-'t Hooft operators [8, 29]. As mentioned above, to further substantiate our conjecture, one should study the full moduli space of solutions to the localization equations of [21] in the presence of the monopole singularity, and rigorously derive the resulting 2d theory. Working out the details and derive new results for Wilson-'t Hooft correlators for most general gauge groups and choice of representations would also be a natural step in which to test our proposals. Finally, one may include in the story also other physical observables of the 4d theory for which the same localization frameworks applies, for example the local chiral primary operators on S^2 studied in [26][3]. By allowing field configurations which are singular on the interesting S^2, one

[2] In a minuscule representation all weights have the same length. For $G = U(N)$, the minuscule representations are the totally antisymmetric representations of arbitrary rank.

[3] The action of S-duality on local operators has been studied in [30, 31], and a perturbative calcu-

should also be able to include in the same setup supersymmetric surface operators [33, 34, 35]. This would hopefully give a rich array of new exact results in $\mathcal{N}=4$ SYM which may be used to further our understanding of the S-duality symmetry, and may be also useful in the context of the holographic duality to string theory in $AdS_5 \times S^5$. Recently, interesting works on loop and surface operators in supersymmetric gauge theories have appeared [36, 37], which used the relation between 4d $\mathcal{N}=2$ gauge theories and Liouville theory uncovered in [38]. Another recent work [25] relates 4d susy gauge theory to 2d gauge theory and integrable systems. It would be interesting to find connections between those papers and the present work.

The paper is organized as follows. In Section 2 we set up our notations and conventions. In Section 3 we briefly review the classical abelian electric-magnetic duality. In Section 4 we give a general definition of locally BPS 't Hooft loop operators in $\mathcal{N}=4$ SYM supported on arbitrary contours, and we evaluate their expectation value in the semiclassical limit. In particular we discuss regularization and introduce the relevant boundary term which makes the computation finite. In Section 5 we review the basic steps of the localization calculation of [21], show that the circular 1/2 BPS 't Hooft loop solves the relevant supersymmetry equations and state our main conjecture that relates the 't Hooft loops to the unstable instantons of the 2d theory. In Section 6 and 7 we apply our conjecture to derive respectively the 't Hooft loop expectation value and the Wilson-'t Hooft correlator from 2d YM.

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2 Conventions

We consider Yang-Mills gauge theory with gauge group G in Euclidean signature. Let $\mathfrak{g}$ denote the Lie algebra of G. Our convention is that the gauge field A takes value in $\mathfrak{g}$, e.g. in the anti-hermitian matrices if $G=U(N)$. By $D_\mu = \partial_\mu + A_\mu$ we denote the covariant derivative. The curvature is the two-form $F = dA + A\wedge A$, i.e. $F_{\mu\nu} = [D_\mu, D_\nu]$.

lation of the local operator-'t Hooft loop correlator has recently appeared in [32].

In the usual physics notations $D_\mu = \partial_\mu - iA'_\mu$ and $F'_{\mu\nu} = i[D_\mu, D_\nu]$, where A' and F' are represented by Hermitian matrices. So we have the relations $A = -iA'$ and $F = -iF'$.

The Yang-Mills functional is

$$S_{YM} = -\frac{1}{g_{4d}^2}\int \operatorname{tr} F \wedge *F - \frac{i\theta}{8\pi^2}\int \operatorname{tr} F \wedge F\,, \tag{2.1}$$

which depends on two real coupling constants g_{4d} and θ which we combine into a complex coupling constant

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{4d}^2}\,. \tag{2.2}$$

In the equation (2.1) the symbol tr for $U(N)$ gauge group is the trace in the fundamental representation. Notice that the bilinear form $\operatorname{tr}(\mathfrak{g}, \mathfrak{g})$ is negatively defined, so the first term in S_{YM} (2.1) is positive. In coordinate notations we have $\int F \wedge *F = \frac{1}{2}\int \sqrt{g} F_{\mu\nu}F^{\mu\nu}$ and $\int F \wedge F = \frac{1}{2}\int \sqrt{g}\varepsilon_{\mu\nu\rho\lambda}F^{\mu\nu}F^{\rho\lambda}$.

Our Lie algebra conventions are the following. By T_a we denote the generators, or basis elements of $\mathfrak{g}$, so for $A \in \mathfrak{g}$ we write $A = A^a T_a$, and we take the coordinates A^a to be real. We choose metric (Killing form) on $\mathfrak{g}$ such that the short coroot has length 2. For example, for $G = U(N)$ or $G = SU(N)$ the metric $\langle , \rangle$ on $\mathfrak{g}$ is given by minus trace in the fundamental representation $\langle a, b\rangle = -tr_F ab$. In the basis T_a the metric has matrix form $g_{ab} = -\operatorname{tr}_F T_a T_b$. We use this metric to raise and lower Lie algebra indices. The second Casimir operator (Laplacian on group G) is defined as $C_2 = -T^a T_a = -g^{ab}T_a T_b$, and C_2 has positive eigenvalues. It is eigenvalue in representation R is called $C_2(R)$, explicitly $-g^{ab}R(T_a)R(T_b) = C_2(R)1_{d_R \times d_R}$. For example, for $U(N)$ we have $C_2(F) = N$ and for $SU(N)$ we have $C_2(F) = \frac{N^2-1}{N}$.

3 Elementary review of abelian S-duality

Electric charge

Given a contour C and a representation R of G we define

$$W_R(C) = \operatorname{tr}_R \operatorname{Pexp}\oint A_\mu dx^\mu = \operatorname{tr}_R \operatorname{P}e^{-i\oint A'}. \tag{3.1}$$

Then the partition function in the presence of the Wilson loop C is

$$\langle W_R(C)\rangle = \frac{1}{Z}\int [\mathcal{D}A] e^{-S_{YM}} \operatorname{tr}_R \operatorname{P}e^{-i\oint A'}. \tag{3.2}$$

Consider the abelian theory $G = U(1)^r$ with coupling constant g_{4d}. We take $r = 1$ for brevity and we set $\theta = 0$. It will be elementary to generalize to arbitrary r later.

The $U(1)$ representations are labeled by an integer $R = n \in \mathbb{Z}$, and $\operatorname{tr}_n e^{i\alpha} := e^{ni\alpha}$. To compute (3.2) classically we need to find the critical point of the exponent in (3.2)

$$S_{YM}[A'] + in \oint A'. \tag{3.3}$$

Of course, we get the usual Maxwell equations with the source

$$-\frac{2}{g_{4d}^2} D_\mu F'_{\mu\nu} + in J_\nu = 0\,, \tag{3.4}$$

where J_ν is the source supported on the contour C.

Let C to be the straight line in the direction x_0 located at the origin $x_1 = x_2 = x_3 = 0$. Solving (3.4) we get the usual Coulomb law

$$A'_0 = -\frac{i g_{4d}^2 n}{8\pi r}, \quad E'_r = F'_{0r} = \frac{i g_{4d}^2 n}{8\pi r^2}\frac{x_i}{r}. \tag{3.5}$$

Here r is the 3d distance from the origin and $E_r = F_{0r} = F_{0i}\frac{x^i}{r}$ is the radial component of the electric field strength.

The energy of the point electric charge diverges. To regularize it, we introduce a UV cut-off small distance ε and delete a solid tube of radius ε surrounding the contour C.

The contribution of the action term in (3.3), evaluated per unit time, gives

$$S_{YM}[A^{el}_{cl}, n] = \frac{1}{g_{4d}^2}\int_\varepsilon^\infty 4\pi r^2 dr (F'_{0r})^2 = -\frac{n^2 g_{4d}^2}{16\pi\varepsilon}\,. \tag{3.6}$$

The contribution of the source term $in \oint A'$ is

$$in A'_{cl}|_{r=\varepsilon} = \frac{g_{4d}^2 n^2}{8\pi\varepsilon}\,. \tag{3.7}$$

As a result we get that classical regularized vev of the Wilson loop per unit length is computed as $e^{-\mathcal{E}_{elec}}$ where

$$\mathcal{E}_{elec} = S_{YM}[A_{cl}] + in A'_{cl} = \frac{g_{4d}^2 n^2}{16\pi\varepsilon}\,. \tag{3.8}$$

Magnetic charge

Now we consider 't Hooft loop operator in the $U(1)$ theory with coupling constant g_{4d}. The classical Maxwell equations in the absence of source terms are invariant under exchange of electric and magnetic fields. Hence we find the field strength associated to the magnetic charge is given by

$$\begin{aligned} F'_{0j} &= 0 \\ F'_{jk} &= \frac{m}{2}\varepsilon_{ijk}\frac{x_i}{r^3}, \end{aligned} \tag{3.9}$$

where m is yet an arbitrary constant and the factor $1/2$ is introduced for convenience. The constant m is quantized, of course, as can be easily seen by integrating the two form F over a spherical surface S^2 surrounding the magnetic charge. One can see that m has the meaning of the first Chern class for the gauge bundle restricted to S^2

$$m = \frac{i}{2\pi}\int_{S^2} F, \tag{3.10}$$

and hence m is an arbitrary integer. This integer is the magnetic charge of the 't Hooft loop operator.

The action for the magnetic charge diverges like in the case of the electric charge. To compute the regularized action we integrate outside the tubular neighborhood of radius w surrounding the magnetic charge. Then we get

$$\mathcal{E}_{mag} = S_{YM}[A^{mag}_{cl}, m] = \frac{1}{g^2_{4d}}\int_w^\infty 4\pi r^2 dr (F')^2 = \frac{\pi m^2}{g^2_{4d}}\,. \tag{3.11}$$

Abelian S-duality

Under S-duality the Wilson loop is mapped to the 't Hooft loop while the coupling constants are related as

$$g^2_{4d} \mapsto \frac{16\pi^2}{g^2_{4d}}. \tag{3.12}$$

Clearly, the energies (3.6) and (3.11) coincide under (3.12) and $n \mapsto m$.

4 Locally BPS 't Hooft operator

In the $\mathcal{N} = 4$ Yang-Mills it is customary to study Wilson loop operators coupled to scalar fields [39]

$$W_R(C) = \frac{1}{d_R}\,\mathrm{tr}_R\,\mathrm{Pexp}\oint A_\mu dx^\mu + i\theta^A(s)\Phi_A ds. \tag{4.1}$$

Here $\theta^A(s)$ specifies couplings to the scalar fields Φ_A, $A = 1\dots 6$, of $\mathcal{N} = 4$ SYM theory. If $\theta^A\theta^A = 1$ then the operator (4.1) is called locally BPS, because for any point x on the contour C one can find 8 supercharges $Q_\alpha(x)$ which locally annihilate Wilson loop (4.1) at the point x. For special choices of contour and of $\theta^A(s)$ one can obtain operators which globally preserve some supercharges [13, 40]. The well known 1/2 BPS case is obtained by $\theta^A(s) = const$, and by taking the contour to be a circle (or a straight line).

It is also elementary to check that in the leading order of perturbation theory the Wilson loop (4.1) on smooth contour C is finite if and only if $\theta^A(s)\theta^A(s) = 1$. The propagators for the gauge and scalar fields on $\mathbb{R}^4$ are (we choose Feynman gauge $\partial_\mu A_\mu = 0$)

$$\begin{aligned} G^{ab}_{\mu\nu} &:= \langle A^a_\mu A^b_\nu \rangle = \frac{g^2_{4d}}{8\pi^2}\frac{g^{ab}g_{\mu\nu}}{(x-y)^2} \\ G^{ab}_{AB} &:= \langle \Phi^a_A \Phi^b_B \rangle = \frac{g^2_{4d}}{8\pi^2}\frac{g^{ab}\delta_{AB}}{(x-y)^2}. \end{aligned} \tag{4.2}$$

Then in the leading order we get

$$\langle W_R(C)\rangle = 1 - \frac{g^2_{4d}}{16\pi^2}C_2(R)G^{(2)}(C)\,, \tag{4.3}$$

where we have denoted

$$G^{(2)}(C) = \oint ds \oint ds' (\frac{\dot{x}^\mu(s)\dot{x}_\mu(s') - \theta^A(s)\theta_A(s')}{(x(s)-x(s'))^2}). \tag{4.4}$$

We notice that the contour shape dependent functional $G^2(C)$ is negative and conformally invariant. In the case of circle and constant θ^A we get $G^{(2)}(C) = -2\pi^2$. In the abelian case, for $G = U(1)^r$, it is elementary to compute exact expectation value of the Wilson loop (4.1) because the path integral is Gaussian. The Wilson loop (4.1) reduces to

$$W_R(C)^{abel} = \frac{1}{d_R}\sum_{\alpha\in\mathrm{irreps}(R)} e^{iw^\alpha_a \oint A^a_\mu dx^\mu + i\theta^A(s)\Phi^a_A ds}. \tag{4.5}$$

Here index α runs over irreducible representations in the decomposition $R = \sum_\alpha R_\alpha$. Each irreducible representation of abelian group G is one-dimensional and is defined by its weight w, which is a one-form on $\mathfrak{g}$, by the rule that the an element of G of the form $e^{A^a T_a}$ is represented by a complex number $e^{iA^a w_a}$. Computing the Gaussian integral with insertion (4.5) we get

$$\begin{aligned} \langle W_R(C)^{abel}\rangle &= \frac{1}{d_R}\sum_{\alpha\in\mathrm{weights}(R)} e^{-\frac{1}{2}\langle w_a \oint (A^a_\mu dx^\mu + i\theta^A(s)\Phi^a_A ds)(w_b \oint A^b_\nu dx^\nu + i\theta^B(s)\Phi^b_B ds)\rangle} = \\ &= \frac{1}{d_R}\sum_{\alpha\in\mathrm{weights}(R)} e^{-\frac{g^2_{4d}}{16\pi^2}\langle w^\alpha, w^\alpha\rangle G^{(2)}(C)}. \end{aligned} \tag{4.6}$$

Irreducible representations of $G = U(1)^r$ are labeled by r-dimensional integer vector $\vec{n} \in \mathbb{Z}^r$. If metric on $\mathfrak{g}$ is fixed as minus trace in the fundamental representation then $\langle w, w \rangle = \vec{n}^2$ for weight w associated to representation $\vec{n}$.

The Wilson loop operator (associated to an electric charge) is the usual operator defined as a functional on the space of fields. To compute expectation value (or correlation functions) for Wilson loop operator, one just insert the corresponding functional under the sign of the path integral. On the other hand, the 't Hooft operator (associated to magnetic charge, or monopole) is a disorder operator, defined by a prescribed singularity for the fields [5, 8]. To compute expectation value or correlation functions for 't Hooft operator, one actually changes the definition of the path integral itself. Instead of integrating over arbitrary smooth fields on space-time, we require the fields to be smooth everywhere except at the location of the disorder operator, where the fields are required to have the prescribed singular behavior.

More concretely, the 't Hooft operator is defined as follows [5, 7, 8]. For the gauge group G we choose group homomorphism $\rho : U(1) \to G$. Such homomorphisms ρ are labeled by the coweights of G, or, equivalently, by the weights of the dual group ${}^L G$. Given ρ and the contour C the 't Hooft operator is defined by asking the gauge fields to have singularity near C like the image under ρ of the basic $U(1)$ monopole (3.9).

We are particularly interested in the partially BPS supersymmetric 't Hooft loops in the $\mathcal{N} = 4$ super Yang-Mills. Similarly to the supersymmetric Wilson loop operator, which couples to the scalar fields in the $\mathcal{N} = 4$ super Yang-Mills, we also turn on coupling to the scalar fields for the supersymmetric 't Hooft operator.

Now we define locally BPS 't Hooft loop operator by generalizing the definition of 1/2 BPS 't Hooft loop operator supported on a straight line [8] (here we consider the $\mathrm{Re}\tau = 0$ case). Given a smooth, not self-intersecting contour C and smooth couplings $\theta^A(s)$, $s \in C$ such that $\theta(s)^2 = 1$, we require that the gauge field and the scalar field have the following singularity in the neighborhood of C

$$
\begin{aligned}
F_{kl}(y) &= \frac{1}{2}\varepsilon_{ijkl}\frac{dx^i}{ds}\frac{(y_j - x_j)}{|y-x|^3}T_{\vec{m}} + O(1) \\
\Phi_A(y) &= \frac{\theta^A(s)}{2|y-x|}T_{\vec{m}} + O(1), \quad \text{in the limit} \quad |y-x| \to 0 \\
T_{\vec{m}} &:= -i\,\mathrm{diag}(m_1, \ldots, m_N)\,,
\end{aligned}
\tag{4.7}
$$

where for each point y in the neighborhood of C, the point $x \in C$ is the point closest to y. If we consider normal hyperplane $\mathbb{R}^3_x$ for each point $x \in C$, the fields F and $\Phi = \Phi_A \theta^A$ approximately satisfy Bogomolny equation [7, 41] in the infinitesimal neighborhood of x

$$
*_{\mathbb{R}^3} F + d\Phi = 0, \tag{4.8}
$$

hence the singularity (4.7) defines a locally BPS 't Hooft operator. The globally supersymmetric 1/2 BPS 't Hooft loop is given by (4.7) with $\theta^A = const$ and C straight line or circle.

The expectation value and correlation functions of 't Hooft operator are defined by taking the path integral over all fields with the asymptotics (4.7). In the semiclassical limit, the main contribution to the path integral is given by the critical point of the action, i.e. by a classical configuration which satisfies the equations of motion and has the required asymptotics (4.7). Clearly, the action evaluated on such configuration will diverge in the region close to the contour C. The difference with the corresponding computation for locally BPS Wilson loop case is that in the Wilson case the divergent contributions coming from the action for gauge field was of the same magnitude but of opposite sign as the contribution coming from the scalar field. In the 't Hooft case both contributions (for the gauge field and for the scalar field) are of the same sign and are not cancelled. This puzzle, which naively seems to violate the S-duality (say in the abelian case, where classical computation is supposed to be exact), is easily resolved by recalling that when we do semiclassical computation in the Wilson case, to the Yang-Mills action evaluated on classical solution we need to add the source term (3.3). Similarly, in the locally BPS Wilson case, we need to add the source term for Φ when we compute classical expectation value. In the 't Hooft case there is no natural source for the magnetic field, and it is not actually needed in order for abelian S-duality to work. Indeed, for gauge fields one can see that $S_{YM}({}^L g_{4d}, {}^L G; F_W^{cl}) + S_{source}(A_W^{cl}) = S_{YM}(g_{4d}, G; F_T^{cl})$, where A_W^{cl}, F_W^{cl} and F_T^{cl} are fields created by Wilson loop or 't Hooft loop respectively. The naive divergence problem of locally BPS 't Hooft loop and the naive disagreement with the dual locally BPS Wilson loop comes actually from the scalar sector, for the simple reason that in the Wilson case we have taken into account contribution of the source term for Φ, but in the 't Hooft case we have not. Moreover since $S_{source}(\Phi^{cl}) = -2S_{YM}(\Phi^{cl})$, just like for the gauge field, we have that $S_{source}(\Phi^{cl}) + S_{YM}(\Phi^{cl}) = -S_{YM}(\Phi^{cl})$. So our conclusion is that the natural way to resolve this puzzle about divergence and mismatch with S-duality is just to add a source term for the field Φ, chosen such that it creates configuration (4.7), to the definition of 't Hooft loop operator.

For computational purposes we try to give the following more detailed definition of locally BPS 't Hooft loop. We try to give a general definition for contour of arbitrary shape and space-time manifold M equipped with arbitrary Riemannian metric. For a smooth not self-intersecting contour C let $D(C, \varepsilon)$ denote a solid tubular neighborhood of the contour C of size ε

$$D(C, \varepsilon) = \{x \in M | \text{distance}(x, C) < \varepsilon\}. \tag{4.9}$$

Then $M(C, \varepsilon) = M \setminus D(C, \varepsilon)$ is a four-dimensional manifold with a boundary. We call

this boundary $\Sigma_3(C,\varepsilon) = \partial D(C,\varepsilon) = -\partial M(C,\varepsilon)$. In the path integral we integrate over all field configurations in the 4d bulk space $M(C,\varepsilon)$. For the gauge field we fix Dirichlet boundary conditions on $\Sigma_3(C,\varepsilon)$ as given by classical configuration which satisfies (4.7).

For the scalar fields we fix Neumann boundary conditions on $\Sigma_3(C,\varepsilon)$ as defined by (4.7), or, equivalently, we can insert source term for the field Φ with support on the boundary $\Sigma_3(C,\varepsilon)$. While specifying boundary conditions for the fields in the form (4.7) we break the gauge group $U(N)$ to $U(1)^r$ on the boundary. In other words, when we factorize the path integral over gauge transformations we require a gauge transformation $g(x)$ to be a smooth G-valued function on $M(C,\varepsilon)$ with boundary conditions on $\Sigma_3(C,\varepsilon)$ specified by restricting $g(x)$ to the maximal torus $T \in G$ for $x \in \Sigma_3(C,\varepsilon)$. For closed contour, the 3d manifold $\Sigma_3(C,\varepsilon)$ has topology $S^1 \times S^2$, and for sufficiently small ε it can be naturally given the structure of foliation. Namely, for each point $s \in C$ define the two-manifold $\Sigma_2(s,C,\varepsilon) \subset \Sigma_3(C,\varepsilon)$ as a set of point in $\Sigma_3(C,\varepsilon)$ which are located at the distance ε from s (and ε is minimal possible distance). Then $\Sigma_3(C,\varepsilon)$ is represented as a S^2-fiber bundle over C, where for each point $s \in C$ the S^2-fiber is $\Sigma_2(s,C,\varepsilon)$. In the following, we will employ the short-hand notation $M(C,\varepsilon) = M_\varepsilon$, $\Sigma_3(C,\varepsilon) = \Sigma_3$, $\Sigma_2(s,C,\varepsilon) = \Sigma_2(s)$. Given the structure of the fiber bundle $\Sigma_2(s) \to \Sigma_3 \to C$, there is a natural coordinate s on Σ_3 induced by length parameter s on C, and the associated one-form ds. Also, the scalar couplings $\theta^A(s)$ could be pulled back on Σ_3 from C.

Given the above geometrical definitions, one natural way to write down the source term for the field Φ is the boundary action on Σ_3 of the form[4]

$$\frac{2}{g_{4d}^2}\,\mathrm{tr}\int_{\Sigma_3} F \wedge \Phi_A \theta^A \wedge ds. \tag{4.10}$$

This boundary action can be interpreted as a source term for the field Φ after we integrate over gauge fields, so that F becomes proportional to the volume form on the S^2 fibers. Such boundary term is natural from the point of view of Bogomolny equations. The YM action coupled to the scalar field Φ on a 3d manifold M_3 with boundary ∂M_3 is the square of the equation (4.8) up to the boundary term

$$-\,\mathrm{tr}\int_{M_3}(*F + D\Phi)\wedge *(*F + D\Phi) = -\,\mathrm{tr}\int_{M_3}(F \wedge *F + D\Phi \wedge *D\Phi) + 2\,\mathrm{tr}\int_{\partial M_3} F \wedge \Phi\,. \tag{4.11}$$

[4]We thank A. Kapustin for a useful discussion.

The total bulk and boundary action is then

$$S_{YM} + S_{boundary} = -\frac{1}{g_{4d}^2} \operatorname{tr} \int_{M(\varepsilon)} F \wedge *F + d\Phi_A \wedge *d\Phi_A + \frac{2}{g_{4d}^2} \int_{\Sigma_3} \operatorname{tr} F \wedge \Phi_A \theta^A \wedge ds, \tag{4.12}$$

and by (4.11) it clearly vanishes per unit length for the 1/2 BPS 't Hooft line.

Now to compute expectation value of locally BPS 't Hooft loop semiclassically it is enough to evaluate the total action (4.12) on a classical configuration with asymptotics (4.7). These classical fields can be easily found. We make the computation on $\mathbb{R}^4$ to make it more transparent. Then we have

$$\begin{aligned} \Phi_A^{cl}(y) &= \frac{T_{\vec{m}}}{2\pi} \oint_C \frac{\theta^A(s)\, ds}{(y - x(s))^2} \\ F_{kl}^{cl} = \frac{1}{2}\varepsilon_{ijkl}(\partial_i b_j^{cl} - \partial_j b_i^{cl}), \quad \text{where} \quad b_i^{cl}(y) &= \frac{T_{\vec{m}}}{2\pi} \oint_C \frac{dx_i}{(y - x(s))^2}\,. \end{aligned} \tag{4.13}$$

Since the configurations (4.13) solves the equations of motion $\Delta\Phi = 0, dF = 0, d{*F} = 0$ in the bulk M_ε, we can evaluate (4.12) integrating by parts and reducing the integral to the boundary Σ_3. Classically for abelian configurations we have

$$\begin{aligned} \int_{M_\varepsilon} F \wedge *F = \int_{M_\varepsilon} db \wedge *db = \int_{M_\varepsilon} d(b \wedge *db) &= -\oint_{\Sigma_3} b \wedge *db \\ \int_{M_\varepsilon} d\Phi \wedge *d\Phi &= -\oint_{\Sigma_3} \Phi \wedge *d\Phi, \end{aligned} \tag{4.14}$$

so

$$S_{YM} + S_{boundary} = \frac{1}{g_{4d}^2}(\operatorname{tr} \oint_{\Sigma_3} b \wedge *_4 db + \Phi \wedge *_4 d\Phi - 2\Phi ds \wedge *_4 db). \tag{4.15}$$

Now we can plug in the classical solution (4.13) into (4.15) and take the limit $\varepsilon \to 0$. In this limit we can average the values of fields b and Φ over each fiber $\Sigma_2(s)$ using that

$$\begin{aligned} \int_{\Sigma_2(s)} *_4 db &= 2\pi T_{\vec{m}} + O(\varepsilon) \\ \int_{\Sigma_2(s)} *_4 d\Phi_A &= 2\pi T_{\vec{m}} \theta_A(s) ds + O(\varepsilon)\, ds, \end{aligned} \tag{4.16}$$

so we get

$$S_{YM} + S_{boundary} = -\frac{2\pi}{g_{4d}^2} \left(\oint_C \operatorname{tr} T_{\vec{m}} b_{cl}(s) - \oint_C \operatorname{tr} T_{\vec{m}} \Phi_{cl}^A(s) \theta_A(s) ds \right) = \frac{1}{g_{4d}^2} \vec{m}^2 G^{(2)}(C), \tag{4.17}$$

and finally, the classical expectation value of locally BPS 't Hooft on arbitrary contour C is given by

$$\langle T_R(C)\rangle = \exp(-(S^{cl}_{YM} + S^{cl}_{boundary})) = \exp(-\frac{1}{g^2_{4d}}\vec{m}^2 G^{(2)}(C))\,. \tag{4.18}$$

We observe that the result for locally BPS 't Hooft loop clearly agrees with the S-dual contribution to the Wilson loop (4.6) under replacement $g^2_{4d} \to \frac{16\pi^2}{g^2_{4d}}$.

5 Localization of 4d $\mathcal{N}=4$ SYM to the 2d theory

We consider the same geometrical setup of [21] where, extending the work [11], it was shown how to use localization in the context of the 1/8 BPS Wilson loops of [12, 13, 14] to obtain from $\mathcal{N}=4$ SYM the two-dimensional theory on S^2, which was called in [21] *almost* 2d Yang-Mills theory. This theory is related to the Yang-Mills-Higgs theory [22, 23, 24, 25].

In [21] a set of supersymmetric equations was derived from the appropriate fermionic symmetry of the Wilson loop operators, and there it was shown that the smooth solutions of these equations are parameterized by two-dimensional data, i.e. by certain field configurations on S^2. It was also mentioned in [26] that within the same setup one can consider the solutions of those supersymmetric equations with singularities which correspond to the insertion of 't Hooft loop operators.

Now we briefly review the construction in [21]. We take the space-time to be the four-sphere S^4, which can be interpreted as the one-point compactification of $\mathbb{R}^4$. Then we represent the S^4 as a warped $S^2 \times S^1$ fibration over an interval I, such that the metric takes the form

$$ds^2 = d\xi^2 + \sin^2\xi(d\theta^2 + \sin^2\theta d\phi^2) + \cos^2\xi d\tau^2. \tag{5.1}$$

The $\xi \in [0, \pi/2]$ is the coordinate on the interval I, the τ is the coordinate on S^1 fiber, and (θ, ϕ) are the usual polar coordinates on the S^2 fiber. At $\xi = 0$ the S^2 fiber shrinks to zero size, at $\xi = \pi/2$ the S^1 fiber shrinks to zero size. The relevant 1/8-BPS Wilson loops studied in [12, 13, 14, 18, 19, 20, 26] are located at the largest S^2 fiber at $\xi = \pi/2$.

The fermionic charge Q used in the localization computation [21] squares to a combination of a $U(1)$ rotation along the S^1 direction τ and a rotation in a $U(1)$ subgroup of the $SO(6)$ R-symmetry of $\mathcal{N}=4$ SYM. By the usual arguments the field theory localizes to the equations $Q\Psi = 0$ where Ψ are fermionic fields of the theory. In the $\mathcal{N}=4$ theory one gets sixteen equations, one for each component of Ψ. Then

it can be seen that nine equations tell us that all fields are covariantly constant along the S^1 fiber. At this step the 4d theory localizes to 3d theory on the S^1 quotient of the S^4. This quotient has topology and the natural metric of the solid three-dimensional ball with boundary, which we denote as D^3. The metric on D^3 is given by the first two terms in (5.1), which is just the metric on a three-dimensional semi-sphere.

To write down the susy equations, it is actually convenient to make a smooth Weyl transformation of the metric on D^3 such that the resulting metric is the standard flat metric on the solid ball. Explicitly, after the rescaling, that 4d metric on the warped fibration $D^3 \times_{\tilde{w}} S^1$ becomes

$$ds^2 = dx_i dx_i + \frac{1}{4}(1-x^2)^2 d\tau^2, \quad i = 2,3,4. \tag{5.2}$$

Here the x^i, $i = 2,3,4$ are the standard flat coordinates on $D^3 = \{x_i \in \mathbb{R} | x_i x_i \leq 1\}$. Because of the conformal symmetry of the equations we are free to do such rescaling.

The remaining seven supersymmetric equations are 3d equations on D^3 for the 3d gauge field and five scalar fields (one of six scalar fields of $\mathcal{N}=4$ SYM does not appear in the 3d equations). The equations are invariant under a diagonal $SO(3)$ subgroup of $SO(3)_{Lorentz} \times SO(3)_R$, where $SO(3)_R$ is a subgroup of the $SO(6)_R$ R-symmetry group. The scalars which transform under $SO(3)_R$ are denoted by Φ_6, Φ_7, Φ_8 (these are the three scalars which couple to the 1/8 BPS Wilson loops [12, 13, 14]). The remaining two scalar fields are labeled as Φ_5 and Φ_9.[5]

Now we quote the relevant 3d equations from [21]

$$\begin{aligned}
-(1-x^2)D_k\Phi_9 - \frac{1}{2}F_{ij}\epsilon_{ijk}(1+x^2) + \frac{1}{2}[\Phi_{i+4},\Phi_{j+4}]\epsilon_{ijp}(\delta_{pk} - x^2\delta_{pk} + 2x_p x_k)& \\
-[\Phi_5,\Phi_{j+4}](\delta_{jk} + x^2\delta_{jk} - 2x_j x_k) + 2\Phi_9 x_k = 0\,,& \\
[\Phi_9,\Phi_{i+4}](\delta_{ik} + x_i x_k - x^2\delta_{ik}) - D_i\Phi_5(\delta_{ik} - x_i x_k + x^2\delta_{ik}) + 2\Phi_5(1-x^2)^{-1}x_k& \\
+D_i\Phi_{j+4}(\epsilon_{ijk} - x_i x_p\epsilon_{jpk} - x_j x_p\epsilon_{ipk}) - 2\Phi_{i+4}\epsilon_{ijk}x_j e_{k+4} = 0\,,& \\
[\Phi_9,\Phi_5](1-x^2) + D_i\Phi_{j+4}(\delta_{ij} + \delta_{ij}x^2 - 2x_i x_j) - 2\Phi_{j+4}x_j = 0\,.&
\end{aligned} \tag{5.3}$$

It is convenient to represent the Φ_{i+4}, $i = 2,3,4$ scalar fields as three components of adjoint valued one-form. Then the gauge field and the adjoint valued one-form can be combined into a complexified connection, while the remaining two scalars can be combined into a complexified scalar. At the origin of D^3 the equations take the form

[5] The fields $\Phi_6, \ldots, \Phi_9$ are exactly the original scalar fields of the $\mathcal{N}=4$ theory, but the field Φ_5 here denotes a twisted combination $\Phi_5 = \sin\tau\Phi'_0 + \cos\tau\Phi'_5$, where Φ'_0 and Φ'_9 stand for the original scalars of $\mathcal{N}=4$ SYM. The orthogonal twisted field $\Phi_0 = -\sin\tau\Phi'_5 + \cos\tau\Phi_0$ does not couple to the 3d equations. In the absence of a θ-angle, and if singular configurations on the S^2 are not allowed, the remaining nine supersymmetry equations are solved by $\Phi_0 = 0$ and $F_{\tau i} = 0$.

of the extended Bogomolny equations [5, 7] which generalize the usual Bogomolny equations by doubling the number of fields

$$\begin{aligned} -*(F-\Phi\wedge\Phi)-d_A\Phi_9+[\Phi,\Phi_5]=0\,,\\ *d_A\Phi-d_A\Phi_5-[\Phi,\Phi_9]=0\,,\\ d_A*\Phi+[\Phi_9,\Phi_5]=0. \end{aligned} \tag{5.4}$$

Here Φ denotes the adjoint valued one-form whose components are Φ_6, Φ_7, Φ_8, and $*$ is the three-dimensional Hodge star.

The equations (5.3) look complicated, however their detailed analysis in the absence of singularities is possible, and one gets the moduli space which is parameterized by certain data on the two-dimensional boundary [21], roughly speaking by the fields of almost 2d Yang-Mills theory.

Now we want to insert a supersymmetric 't Hooft loop running along the S^1 fiber at the origin ($0 \leq \tau < 2\pi$, $x_2 = x_3 = x_4 = 0$). This is equivalent to introducing a singularity at the origin of the prescribed form into the solutions of the equations (5.3). Let us first look at the simplified equations (5.4) close to the origin of D^3. To introduce the conventional BPS monopole singularity, we can actually set to zero the fields (Φ, Φ_5) in (5.4). Then one is left with the classical Bogomolny equation for BPS monopole

$$*F+d_A\Phi_9=0. \tag{5.5}$$

Effectively abelian solutions with singularity at the origin for the $U(N)$ gauge group are easily described by coupling the $U(1)$ monopole (3.9) with the scalar field Φ_9 and picking up a homomorphism $\rho : U(1) \to G$. Explicitly, for ρ represented by N-tuple $(m_1, \ldots, m_N)$ we ask the fields to have singularity near the origin of the form

$$\begin{aligned} F_{jk}=\frac{1}{2}\varepsilon_{ijk}\frac{x_i}{r^3}T_{\vec{m}}\,,\\ \Phi_9=\frac{1}{2r}T_{\vec{m}}, \end{aligned} \tag{5.6}$$

where $r = \sqrt{x_i x_i}$.

After understanding the solution (5.6) for the simplified equations, it is elementary to write down the effectively abelian (breaking $U(N)$ to $U(1)^N$) solution of the full system (5.3) with the same kind of singularity. Namely, just set again to zero the fields $\Phi_5, \ldots, \Phi_8$. Then equations (5.3) are consistently reduced to

$$(1+x^2)\frac{1}{2}F_{ij}\epsilon_{ijk}+D_k(\Phi_9(1-x^2))=0\,. \tag{5.7}$$

Now one can check that the solution (5.6) satisfies the equations (5.7), and hence it also solves the full system (5.3) (provided we keep $\Phi_5, \ldots, \Phi_8 = 0$). Of course, (5.6) is nothing but the singularity associated with a 1/2 BPS circular 't Hooft loop.

So far we have presented just one point on the moduli space of solutions of susy equations (5.3) with a prescribed singularity at the origin. Let us call this moduli space $\mathcal{M}_{\vec{m}}$, and the point corresponding to the solution (5.6) as a reference point $p_{\vec{m}} \in \mathcal{M}_{\vec{m}}$.

To complete the localization analysis, one would like to find the complete moduli space $\mathcal{M}_{\vec{m}}$ and map it to the two-dimensional data on the boundary S^2, similarly to what has been done in [21] for smooth solutions. The four-dimensional path integral is then reduced to the two-dimensional path integral over $\mathcal{M}_{\vec{m}}$, or, equivalently, over the the boundary data on S^2.

We leave the detailed analysis of the equations and of $\mathcal{M}_{\vec{m}}$ for future study. In the present work we just look at point $p_{\vec{m}} \in \mathcal{M}_{\vec{m}}$ from the following perspective. Namely, we observe that the gauge field (5.6) restricted to the boundary sphere S^2 has precisely the form of the (unstable) instanton in the two-dimensional Yang-Mill labeled by N-tuple $m_1, \dots, m_N$. Recalling that in the absence of any singularities the $\mathcal{N} = 4$ SYM four-dimensional path integral has been argued to reduce to the zero-instanton sector of two-dimensional Yang-Mills [21], it is tempting to conjecture that *integration over $\mathcal{M}_{\vec{n}}$ is equivalent to the corresponding (unstable) instanton contribution to the partition function of two-dimensional Yang-Mills.* This is the key conjecture of the present paper which allows us to compute exactly the expectation value of circular BPS 't Hooft operator without reference to S-duality. We do this computation in the next sections using the very well known partition function of 2d Yang-Mills on S^2 and we find explicitly precise agreement with S-duality predictions.

Elementary evaluation of $S_{YM} + S_{boundary}$ on classical solution

On $D^3 \times_{\tilde{w}} S^1$ (see the metric (5.2)) the SYM action [21] evaluated on the classical solution (5.6) gives the integral

$$S_{YM} = -\frac{1}{g_{4d}^2} \int_{M_\varepsilon} \mathrm{tr}\, \sqrt{g} (\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + D_\mu \Phi_A D^\mu \Phi^A + \frac{R}{6} \Phi_A \Phi^A) =$$

$$= \frac{1}{g_{4d}^2} (2\pi)(4\pi) \frac{\vec{m}^2}{4} \int_\varepsilon^1 dx\, x^2 (\frac{1}{x^4} + \frac{1}{x^4} + \frac{2}{1-x^2} \frac{1}{x^2}) \frac{1}{2} (1-x^2) = \frac{2\pi^2 \vec{m}^2}{g_{4d}^2} (\frac{1}{\varepsilon} - 1)\,, \quad (5.8)$$

where we have regularized the integral by cutting out the region $x_i x_i < \epsilon^2$. The boundary term (4.10) evaluated on the resulting 3d boundary gives

$$S_{boundary} = -\frac{2\pi^2 \vec{m}^2}{g_{4d}^2} \frac{1}{\varepsilon}, \quad (5.9)$$

so the total action is

$$S_{YM}(T_{\vec{m}}) + S_{boundary}(T_{\vec{m}}) = -\frac{2\pi^2 \vec{m}^2}{g_{4d}^2}\,. \quad (5.10)$$

Notice that we get exactly the same result as for the 1/2 BPS circular loop on $\mathbb{R}^4$, see eq. (4.18), as expected by conformal invariance.

6 BPS 't Hooft loops from 2d Yang-Mills unstable instantons

Let us consider 2d Yang-Mills theory on S^2 with gauge group $U(N)$. In our conventions, the action reads

$$S = -\frac{1}{2g^2}\int_{S^2} d^2x\sqrt{g}\,\mathrm{tr}\,F^2\,. \tag{6.1}$$

It can be shown [42, 43, 44, 45] that two-dimensional Yang-Mills theory localizes on the classical configurations solving $D * F = 0$, called (unstable) instantons. For $U(N)$ gauge group each such configuration on S^2 is labeled by N integers $m_1, \ldots, m_N$. In the standard polar coordinates, the explicit instanton solutions may be written as the diagonal matrix

$$F_{inst} = \frac{1}{2}\sin\theta d\theta \wedge d\phi\, T_{\vec{m}}\,. \tag{6.2}$$

The exact partition function of 2d YM on S^2 has a representation as a sum over such configurations

$$Z^{YM_2}_{S^2}(g) = \sum_{m_i=-\infty}^{\infty} Z(g; m_1, \ldots, m_N). \tag{6.3}$$

Each instanton configuration contributes with the usual classical weight $Z_{class} = e^{-S_{inst}}$ multiplied by the factor Z_{quant} accounting for the quantum fluctuations [46, 47, 48, 49]

$$Z(g; m_1, \ldots, m_N) = \exp(-S_{inst}(g; m_i))Z_{quant}(g; m_1, \ldots, m_N), \tag{6.4}$$

where

$$S_{inst}(g; m_i) = \frac{4\pi^2}{g^2 A}\sum_{i=1}^{N} m_i^2, \tag{6.5}$$

is the classical action (6.1) evaluated on the instanton solution (6.2), and A is the area of S^2. This agrees with our classical 4d computation (5.10) as supposed under the relation [12, 14, 21]

$$g^2 = -\frac{2g^2_{4d}}{A}. \tag{6.6}$$

The localization arguments discussed in the previous section lead us to propose that the exact expectation value of the 1/2 BPS circular 't Hooft loop in representation ${}^L R = (m_1, \ldots, m_N)$ in $\mathcal{N} = 4$ SYM with gauge group G can be computed from the

partition function of 2d YM with gauge group G around an unstable instanton labeled by ${}^L R$

$$\langle T_{{}^L R}(\mathcal{C})\rangle \leftrightarrow \frac{Z(g; m_1, \ldots, m_N)}{Z(g; 0, \ldots, 0)}\,, \tag{6.7}$$

where the normalization by the 0-instanton partition function is such that the 't Hooft loop in trivial representation has unit expectation value. Actually, because of the phenomenon known as "monopole bubbling" [5], we expect that the "naive" 't Hooft loop corresponding to a single unstable instanton in 2d YM according to (6.7) will give the full exact result only for the case of the rank k antisymmetric representation $R = A_k = (\overbrace{1, \ldots, 1}^{k}, 0, \ldots, 0)$ (including the fundamental as a special case). This is the representation with smallest $\sum_i m_i^2$ for fixed $\sum_i m_i$, and cannot be screened to give rise to subleading saddle points. For this reason, we will specialize to this choice of representation in the following, and leave the study of more general representations for future work.

In the localization context, the quantum factor $Z_{quant}(g; m_1, \ldots, m_N)$ usually has cohomological interpretation [50], and it can be exactly computed by the perturbation theory in the coupling constant g. The perturbative series actually terminates at finite order, so $Z_{quant}(g; m_1, \ldots, m_N)$ turns out to be a polynomial of finite degree in g [42, 43, 44, 45]. However, perhaps a simpler way to obtain the instanton representation (6.3) is to start from the well known expression [51, 52] of the exact partition function of 2d YM as a sum over irreducible representations of the gauge group and perform a certain Poisson resummation (see e.g. [15, 53, 54, 55]). In the following we briefly review this approach, following [15].

The exact partition function of 2d YM on S^2 is given by [51, 52]

$$Z_{S^2}^{YM_2} = \sum_R d_R^2 e^{-\frac{g^2 A}{4} C_2(R)}\,, \tag{6.8}$$

where d_R is the dimension of the representation and $C_2(R)$ is the quadratic Casimir. Irreducible representations of $U(N)$ are labeled in the standard way by Young diagrams $\vec{\lambda} = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ where λ_k denote the lengths of rows. The character $\chi_{\vec{\lambda}}(\theta)$, which is defined as trace in representation $\vec{\lambda}$ of the group element $\mathrm{diag}(z_i) \in G$, where $z_i = e^{i\theta_i}$, is given by the Schur polynomial of z_i

$$\chi_{\vec{\lambda}}(e^{i\theta}) = \frac{\det_{ij} e^{i\theta_i l_j}}{\det_{ij} e^{i\theta_i (N-j)}}, \quad l_j = \lambda_j + N - j, \quad i, j = 1, \ldots, N\,. \tag{6.9}$$

Equivalently, irreducible representations of $U(N)$ are labeled by strictly decreasing N-tuples of integers $\infty > l_1 > l_2 > \cdots > l_N > -\infty$, with the character being given by

the same formula (6.9), and the dimension computed as

$$d_\lambda = \frac{\Delta(l_1, \ldots, l_N)}{\Delta(N, \ldots, 1)} = \frac{\prod_{i<j}(\lambda_i - \lambda_j + j - i)}{\prod_{i<j}(j-i)}, \tag{6.10}$$

where Δ denotes the Vandermonde determinant

$$\Delta(l_1, \ldots, l_N) = \prod_{i<j=1}^{N} (l_i - l_j)\,. \tag{6.11}$$

The quadratic casimir $C_2(R)$ for $U(N)$ is

$$C_2(R) = -\frac{N}{12}(N^2-1) + \sum_{i=1}^{N}(l_i - \frac{N-1}{2})^2. \tag{6.12}$$

Then (6.8) can be written explicitly as

$$Z_{S^2}^{YM_2} = \frac{1}{N!}c(N,g) \sum_{l_i=-\infty}^{\infty} \Delta^2(l_1, \ldots, l_N) e^{-\frac{g^2 A}{4}\sum_{i=1}^{N}(l_i - \frac{N-1}{2})^2}, \tag{6.13}$$

where using antisymmetry of Δ with with respect to permutations of l_i we extended the range of summation to arbitrary N-tuples of $l_i \in \mathbb{Z}$, and by $c(N,g)$ we denoted the trivial factor

$$c(N,g) = \frac{1}{(\prod_{i=1}^{N-1} i!)^2} e^{\frac{g^2 A N(N^2-1)}{48}}. \tag{6.14}$$

We now perform a Poisson resummation of (6.13), using the formula

$$\sum_{l_i=-\infty}^{\infty} f(l_1, \ldots, l_N) = \sum_{m_i=-\infty}^{\infty} \int_{-\infty}^{\infty} dz_1 \ldots dz_N e^{2\pi i \sum_{i=1}^{N} m_i z_i} f(z_1, \ldots, z_N)\,. \tag{6.15}$$

Therefore we have, after a simple shift in the z-variable

$$Z_{S^2}^{YM_2} = c(N,g)\frac{1}{N!} \sum_{m_i=-\infty}^{\infty} e^{i\pi(N-1)\sum_i m_i} \int d^N z \prod_{i<j}(z_i - z_j)^2 e^{-\frac{g^2 A}{4}\sum_{i=1}^{N} z_i^2} e^{2\pi i \sum_{i=1}^{N} m_i z_i}\,, \tag{6.16}$$

Each term in the sum over the m_i's is now physically interpreted as the contribution of an unstable instanton (6.2) with quantum numbers $(m_1, \ldots, m_N)$. To make the interpretation more transparent, one can notice that after elementary manipulations we can indeed rewrite (6.16) in the form (6.3)-(6.4)

$$Z_{S^2}^{YM_2} = \sum_{m_i=-\infty}^{\infty} e^{-\frac{4\pi^2}{g^2 A}\sum_{i=1}^{N} m_i^2} Z_{quant}(g; m_1, \ldots, m_N)\,, \tag{6.17}$$

where

$$Z_{quant}(g; m_1, \ldots, m_N) = \frac{1}{N!} c(N, g) e^{i\pi(N-1)\sum_i m_i} \int d^N z \, \Delta^2 \left(\vec{z} + \frac{4\pi i \vec{m}}{g^2 A} \right) e^{-\frac{g^2 A}{4} \sum_{i=1}^N z_i^2} \tag{6.18}$$

corresponds to quantum fluctuations around the unstable instanton.

We now show that our proposal (6.7) is precisely consistent with the S-duality symmetry of $\mathcal{N} = 4$ SYM exchanging magnetic and electric loops. Recall that the expectation value of a circular 1/2 BPS Wilson loop in representation ${}^L R$ in $\mathcal{N} = 4$ SYM with gauge group ${}^L G$ and coupling constant ${}^L g_{4d}$ is computed exactly by a matrix integral over the Lie algebra ${}^L \mathfrak{g}$ of ${}^L G$ [11]. In the case of ${}^L G = G = U(N)$, this is the familiar Gaussian Hermitian matrix model [9, 10]

$$\langle W_{{}^L R}(\mathcal{C}) \rangle = \frac{1}{\mathcal{Z}({}^L g_{4d}^2)} \int DX \, e^{-\frac{2}{{}^L g_{4d}^2} \operatorname{tr} X^2} \, \frac{1}{d_{{}^L R}} \operatorname{tr}_{{}^L R} e^X \,, \tag{6.19}$$

where g_{4d} is the SYM coupling constant, and $\mathcal{Z}({}^L g_{4d}^2)$ is the matrix model partition function. According to S-duality, in the $U(N)$ theory the Wilson loop at coupling ${}^L g_{4d}$ is mapped to the 't Hooft loop at the dual coupling $g_{4d}^2 = 16\pi^2 / {}^L g_{4d}^2$

$$\langle W_{{}^L R}(\mathcal{C}) \rangle_{{}^L g_{4d}} = \langle T_{{}^L R}(\mathcal{C}) \rangle_{g_{4d}} \,. \tag{6.20}$$

The relation of the unstable instanton partition function to the Gaussian matrix model can be quickly recognized by looking at eq. (6.16). Making a simple change of variables $2\pi i z_i = x_i$ and plugging in the map $g^2 A = -2 g_{4d}^2$ between 2d and 4d couplings [13, 14], we have (dropping overall constants which do not depend on the m_i's)

$$Z(g; m_1, \ldots, m_N) \simeq e^{i\pi(N-1)\sum_i m_i} \int d^N x \prod_{i<j} (x_i - x_j)^2 e^{-\frac{g_{4d}^2}{8\pi^2} \sum_{i=1}^N x_i^2} e^{\sum_i m_i x_i} \,. \tag{6.21}$$

The integration over the x_i variables is clearly equivalent to the integration over eigenvalues in a Gaussian Hermitian matrix model with potential $V(X) = \frac{g_{4d}^2}{8\pi^2} \operatorname{tr} X^2$. Moreover, specializing to the rank k antisymmetric representation, one can see that the insertion of $e^{\sum_i m_i x_i} = e^{\sum_{i=1}^k x_i}$ is equivalent to the insertion of the character $\frac{1}{d_{A_k}} \operatorname{tr}_{A_k} e^X$ in the matrix model, since in the eigenvalue basis

$$\frac{1}{d_{A_k}} \operatorname{tr}_{A_k} e^X = \frac{(N-k)! k!}{N!} \sum_{i_1 < i_2 < \cdots < i_k} e^{x_{i_1} + x_{i_2} + \ldots + x_{i_k}} \,. \tag{6.22}$$

But the integrand is symmetric under permutations of the x_i's, hence we can just take one term in the sum above multiplied by d_{A_k}, and we exactly end up with the insertion

of $e^{\sum_{i=1}^{k} x_i}$ in the eigenvalue integral. Putting everything together, the identification (6.7) implies the exact prediction for the 't Hooft loop expectation value

$$\langle T_{A_k}(\mathcal{C})\rangle = \frac{(-1)^{k(N-1)}}{\mathcal{Z}({}^L g_{4d}^2)} \int DX\, e^{-\frac{2}{{}^L g_{4d}^2}\operatorname{tr} X^2} \frac{1}{d_{A_k}} \operatorname{tr}_{A_k} e^X\,, \qquad {}^L g_{4d}^2 = \frac{16\pi^2}{g_{4d}^2}\,. \tag{6.23}$$

This is precisely equal to the Wilson loop expectation value (6.19) in the same representation and with dual coupling constant, up to the overall sign. Fixing this sign requires a more careful study of the one-loop determinant for fluctuations around the supersymmetric configurations. Notice that the correct form of the S-dual coupling ${}^L g_{4d}^2$, including the numerical factor, is correctly predicted by the 2d YM unstable instanton partition function.

As an example, in the case of the fundamental representation we can compute the integral over eigenvalues explicitly by using orthogonal polynomials (see e.g. [10]), and we get the exact result

$$\langle T_F(\mathcal{C})\rangle_{g_{4d}} = \frac{(-1)^{(N-1)}}{N} L_{N-1}^1\left(-\frac{4\pi^2}{g_{4d}^2}\right) e^{\frac{2\pi^2}{g_{4d}^2}}\,, \tag{6.24}$$

where $L_{N-1}^1(x)$ is a Laguerre polynomial[6]. From the point of view of the 2d YM instanton partition function, the exponential factor corresponds to the classical action (6.5), while the Laguerre polynomial comes from the quantum corrections around the instanton.

7 Wilson-'t Hooft correlator

According to our conjecture, we can also compute correlation functions of the 1/2 BPS 't Hooft loop with any number of 1/8 BPS Wilson loops inserted on the S^2 linked to the 't Hooft loop. This is simply done in the 2d theory by calculating the Wilson loop correlation functions around a fixed unstable instanton. As an example, we compute here the correlator of the 't Hooft loop and a Wilson loop in the case in which both operators are in the fundamental representation of $U(N)$. We leave the study of more general representations (and gauge groups) to future study.

Let us start from the exact expression for the expectation value of a Wilson loop in the 2d Yang-Mills theory on S^2 [51, 52]

$$\langle W_R(A_1, A_2)\rangle_{S^2}^{YM_2} = \int dU \sum_{R_1,R_2} d_{R_1} d_{R_2} \chi_{R_1}(U)\bar{\chi}_{R_2}(U) e^{-\frac{g^2 A_1}{4}C_2(R_1) - \frac{g^2 A_2}{4}C_2(R_2)} \frac{1}{d_R}\operatorname{tr}_R U\,, \tag{7.1}$$

[6]The Laguerre polynomials can be defined by $L_n^k(x) = \frac{x^{-k}e^x}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^{n+k}\right)$. We also denote $L_n^0(x) \equiv L_n(x)$.

where the integral is taken over the $U(N)$ group manifold and $\chi_{R_i}(U)$ denotes the character of U in representation R_i. Here A_1, A_2 are the areas of the two regions singled out by the loop on S^2.

Specializing to the case of Wilson loop in the fundamental, after performing the integration over $U(N)$, this can be written explicitly as [15]

$$\begin{aligned}\langle W_F(A_1,A_2)\rangle_{S^2}^{YM_2} = c(N,g)\frac{1}{N!}\sum_{k=1}^{N}\sum_{l_i=-\infty}^{\infty}\Delta(l_i)\Delta(l_i+\delta_{ik})\times \\ \times\, e^{-\frac{g^2A_1}{4}\sum_{i=1}^{N}(l_i-\frac{N-1}{2})^2-\frac{g^2A_2}{4}\sum_{i=1}^{N}(l_i-\frac{N-1}{2}+\delta_{ik})^2}\,.\end{aligned} \tag{7.2}$$

To obtain the instanton expansion of this result, we again perform a Poisson resummation using (6.15), and we get

$$\begin{aligned}&\langle W_F(A_1,A_2)\rangle_{S^2}^{YM_2} = \sum_{m_i=-\infty}^{\infty}\langle W_F(A_1,A_2)\rangle_{(\vec m)}\,,\\ &\langle W_F(A_1,A_2)\rangle_{(\vec m)} = c(N,g)\frac{1}{N!}e^{i\pi(N-1)\sum_i m_i}\times\\ &\qquad\times\sum_{k=1}^{N}\int d^Nz\Delta(z_i)\Delta(z_i+\delta_{ik})e^{2\pi i\sum_i m_i z_i}e^{-\frac{g^2A_1}{4}\sum_{i=1}^{N}z_i^2-\frac{g^2A_2}{4}\sum_{i=1}^{N}(z_i+\delta_{ik})^2}\,.\end{aligned} \tag{7.3}$$

In this formula, $\langle W_F(A_1,A_2)\rangle_{(\vec m)}$ corresponds to the Wilson loop average around an unstable instanton with quantum numbers $\vec m = (m_1,\ldots,m_N)$, and hence, according to our conjecture, it gives the 4d correlator between the 1/8 BPS Wilson loop and the 1/2 BPS 't Hooft loop labeled by $\vec m$[7].

As an example, we now evaluate explicitly the integral in (7.3) in the case of the instanton/'t Hooft loop in the fundamental $\vec m = (1,0,\ldots,0)$. To simplify the equations, we will also restrict in the following to the 1/2 BPS great circle with $A_1 = A_2 = A/2$, but the generalization to arbitrary area is straightforward.

Due to the symmetries of the integrand, the sum over k in (7.3) can be reduced to two terms: the one with $k=1$ and the one e.g. with $k=2$ counted $N-1$ times. The integrals can be performed explicitly using the standard trick of rewriting the Vandermonde determinants in terms of orthogonal polynomials (in the present case it is convenient to use Hermite polynomials, due to the Gaussian integration measure). In evaluating the integrals, the following identity involving Hermite polynomials[8] turns

[7]Modulo the issue of monopole bubbling discussed above, i.e. we expect the naive equivalence to be exact only for 't Hooft loop in the "minuscule" representations.

[8]The Hermite polynomial are given by $H_k(x) = (-1)^k e^{x^2}\frac{d^k}{dx^k}e^{-x^2}$. The formula (7.4) can be proven for example by using the identity $H_k(x+a) = (H+2a)^k$, where it is understood that $H^k \equiv H_k(x)$.

out to be useful

$$\int_{-\infty}^{\infty} dx e^{-x^2} H_k(x+a) H_l(x+b) = 2^k \sqrt{\pi} k! (2b)^{l-k} L_k^{l-k}(-2ab)\,, \quad k \le l\,, \tag{7.4}$$

where $L_k^{l-k}(-2ab)$ is a Laguerre polynomial.

After normalizing $\langle W_F(A/2, A/2)\rangle_{(1,0,\ldots,0)}$ by the 0-instanton partition function, we finally obtain our prediction for the exact correlator of the 1/2 BPS 't Hooft loop and the 1/2 BPS Wilson loop in the 4d $\mathcal{N}=4$ SYM theory

$$\begin{aligned}
\langle T_F(\mathcal{C}) W_F(\mathcal{C}')\rangle &= (-1)^{N-1} \frac{e^{\frac{g_{4d}^2}{8} + \frac{2\pi^2}{g_{4d}^2}}}{N^2} \Bigg\{ - L_{N-1}^1\left(-\frac{g_{4d}^2}{4} - \frac{4\pi^2}{g_{4d}^2}\right) \\
&+ L_{N-1}^1\left(-\frac{g_{4d}^2}{4}\right) L_{N-1}^1\left(-\frac{4\pi^2}{g_{4d}^2}\right) - \sum_{j=1}^N L_{j-1}\left(-\frac{g_{4d}^2}{4}\right) L_{j-1}\left(-\frac{4\pi^2}{g_{4d}^2}\right) \\
&- \sum_{j_1<j_2=1}^N \frac{(j_1-1)!}{(j_2-1)!} \left[(i\pi)^{j_2-j_1} + (-i\pi)^{j_2-j_1}\right] L_{j_1-1}^{j_2-j_1}\left(-\frac{g_{4d}^2}{4}\right) L_{j_1-1}^{j_2-j_1}\left(-\frac{4\pi^2}{g_{4d}^2}\right) \Bigg\}\,,
\end{aligned} \tag{7.5}$$

where $\mathcal{C}'$ denotes the circle at the equator of S^2. Notice that the correlation function is invariant under the S-dual replacement $g_{4d}^2 \to {}^L g_{4d}^2 = 16\pi^2/g_{4d}^2$

$$\langle T_F(\mathcal{C}) W_F(\mathcal{C}')\rangle_{g_{4d}^2} = \langle T_F(\mathcal{C}) W_F(\mathcal{C}')\rangle_{{}^L g_{4d}^2}\,. \tag{7.6}$$

This is precisely as expected, since S-duality exchanges the roles of Wilson and 't Hooft loop in the correlation function[9].

For future reference, we also quote here the small coupling expansion of the result

$$\langle T_F(\mathcal{C}) W_F(\mathcal{C}')\rangle = (-1)^{N-1} e^{\frac{2\pi^2}{g_{4d}^2}} \left(\frac{4\pi^2}{g_{4d}^2}\right)^{N-1} \frac{(N-2)}{N^2(N-1)!} \left[1 + \frac{g_{4d}^2 N}{8\pi^2}(\pi^2 + 2(N-1)) + \ldots\right]. \tag{7.7}$$

We notice that the Wilson loop in the 't Hooft loop background normalized by the expectation value of the 't Hooft loop has perturbative expansion

$$\frac{\langle T_F(\mathcal{C}) W_F(\mathcal{C}')\rangle}{\langle T_F(\mathcal{C})\rangle} = \frac{N-2}{N} + (N-2) g_{4d}^2 + \ldots. \tag{7.8}$$

It it easy to check that (7.8) agrees with the $\mathcal{N}=4$ SYM perturbation theory for W in the background of T. The first term in (7.8) is obtained as a classical value of $W(\mathcal{C}')$

[9]Of course, S-duality does not map $T_F(\mathcal{C})$ and $W_F(\mathcal{C}')$ to each other, but it maps the 't Hooft loop to a Wilson loop on the same circle and vice-versa. But the correlation function is insensitive to whether we put the Wilson/'t Hooft loop on $\mathcal{C}/\mathcal{C}'$.

in the background of $T(\mathcal{C})$, and the second term in (7.8) comes from the one-ladder diagram. The background of 't Hooft loop breaks the $U(N)$ to $U(N-1) \times U(1)$. The diagonal $U(1) \times U(1)$ and $U(N-1) \times U(N-1)$ blocks for the propagators of the relevant gauge and scalar fields in this background are unchanged and contribute respectively as $-g_{4d}^2/8$ and $(N-1)g_{4d}^2/8$ to the one-ladder diagram. The correlators of the anti-diagonal blocks $U(1) \times U(N-1)$ do not contribute to the expectation value of $W(\mathcal{C}')$, so we get (7.8).

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